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Structure of twist-grain-boundary–*C* phases

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We study properties of the Renn-Lubensky twist-grain-boundary–C (TGB$_C$) phase, with layer normal rotating in a plane perpendicular to the pitch axis, and the Bordeaux TGB$_C$ phase, with the layer normal rotating on a cone parallel to the pitch axis near the upper critical twist k_{c2} marking the transition to the cholesteric phase. We introduce a generalized model free energy for the smectic-C phase that allows either TGB$_C$ phase to be stable, and we calculate k_{c2} and the order-parameter profile, which shows only modest spatial variation, for both phases.

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Twist-grain-boundary or TGB phases [1,2,4] are phases of layered smectic liquid crystals [3] induced by molecular chirality. They consist of periodically spaced grain boundaries, each composed of an array of parallel dislocations, separating smectic slabs as depicted in Figs. 1 and 2. The layer normals **N** of the slabs rotate in discrete jumps across the grain boundaries. These remarkable phases are the analog in liquid crystals [5] of the Abrikosov flux lattice in superconductors [6] with the complex smectic mass-density-wave amplitude ψ the analog of the superconducting order parameter, dislocations in the grain boundary the analog of vortices, and the chiral coupling constant h induced by molecular chirality the analog of the external magnetic field H.

The simplest TGB phase is the TGB$_A$ phase in which the smectic slabs between grain boundaries have the character of a bulk smectic-A (SmA) phase in which the layer normal **N** and the director **n**, specifying the direction of average molecular alignment, are parallel to each other in a plane perpendicular to the pitch axis along **p**. In TGB$_C$ phases, the smectic slabs have the character of a bulk smectic-C (SmC) with **n** tilted relative to **N**. Two distinct structures for the TGB$_C$ phase immediately come to mind. In the first [7,8], both **N** and **n** rotate in the plane perpendicular to **p** but with a finite angle between them as shown in Fig. 1. We will refer to this as the Renn-Lubensky or RL TGB$_C$ phase. In the second TGB$_C$ phase, discovered [9] and studied by the Bordeaux group and collaborators [10], **n** rotates in the plane perpendicular to **p**, but **N** rotates on a cone with a component parallel to **p** so that **p** does not lie parallel to the smectic layers as shown in Fig. 2. We will refer to this as the Bordeaux or B TGB$_C$ phase. No pure form of the RL phase has been reported, though phases with two-dimensional modulation of the local RL TGB$_C$ structure have been observed [11]. Though the RL TGB$_C$ structure may be unstable with respect to these modulations, we assume here that it can be stable. We will not discuss the TGB$_{C*}$ phase [8] with smectic-C* slabs in which the director rotates in a cone from layer to layer.

Our goal is to study the structure of both the RL and B phases near the upper critical field h_{c2}, where the TGB$_C$ phase becomes unstable with respect to the cholesteric phase in which the smectic order parameter is zero and the director twists in a helical fashion about **p** with a pitch P. We follow closely the procedure developed by Abrikosov [6] in his analysis of the superconducting flux phase near the upper critical field H_{c2} and applied successfully to the TGB$_A$ phase near h_{c2} [1]. Our analysis of the transition to the bordeaux TGB phase is essentially identical to that presented by

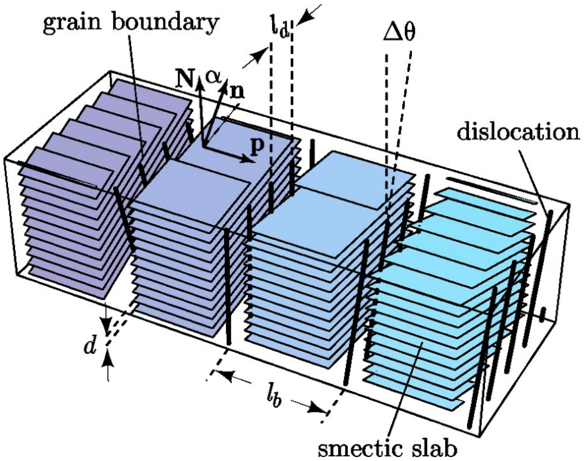

FIG. 1. The RL TGB$_C$ phase. There is a fixed angle between the layer normal **N** and the director **n**, which rotate in the plane perpendicular to the pitch axis **p**.

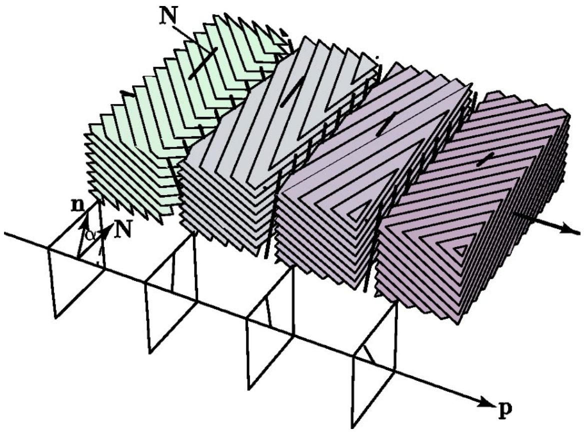

FIG. 2. The Bordeaux TGB$_C$ phase. There is a fixed angle between **N** and **n**, but **n** rotates in the plane perpendicular to **p**, and **N** rotates on a cone whose axis is parallel to **p**.

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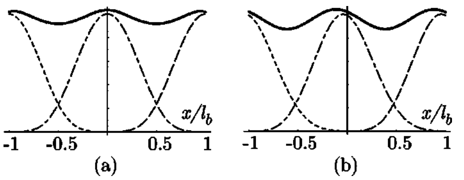
TABLE I. $\mathcal{H}_u$ and proportionality of l_b, l_d, and ξ to powers of P and d in TGB phases.

Phase	$\mathcal{H}_u$	l_b	l_d	ξ
TGB$_A$	$-d^2/du^2+u^2$	$P^{1/2}d^{1/2}$	$P^{1/2}d^{1/2}$	$P^{1/2}d^{1/2}$
Bordeaux	$-d^2/du^2+u^4$	$P^{2/3}d^{1/3}$	$P^{1/3}d^{2/3}$	$P^{2/3}d^{1/3}$
RL	d^4/du^4+u^2	$P^{1/3}d^{2/3}$	$P^{2/3}d^{1/3}$	$P^{2/3}d^{1/3}$

Luk'yanchuk [12]. Using a more general model than his, which does not permit a stable RL phase, we study both the B and RL phases, including their order-parameter profiles, but not the TGB$_{2q}$ phase he introduced.

Several results of our analysis are worthy of note. The linear stability operator associated with the TGB$_A$ phase, like that associated with the Abrikosov phase, is a quantum harmonic oscillator Hamiltonian, $\mathcal{H}_u=-d^2/du^2+u^2$, where u is a rescaled coordinate along $\mathbf{p}$. The same operator associated with the B phase is a u^4-anharmonic oscillator Hamiltonian, $-d^2/du^2+u^4$, whereas that associated with the RL phase is the dual of the Bordeaux operator, d^4/du^4+u^2. The dependence of the grain-boundary spacing l_b, the dislocation spacing l_d within a grain boundary, and the smectic coherence length ξ on the cholesteric pitch P and the layer spacing d is different in the three phases as reviewed in Table I. The near equality of l_b and l_d in the TGB$_A$ phase and their $P^{1/2}d^{1/2}$ dependence on pitch and layer spacing has been verified [2]. Experimentally [10], l_b is substantially larger than l_d in the Bordeaux TGB$_C$ phase in agreement with Table I. Finally, we find that the smectic order parameter, though depressed at the grain boundaries, is reasonably constant in the two TGB$_C$ phases as shown in Fig. 3. We find no evidence in the B phase of melted grain boundaries along which $\psi\sim0$ as suggested by Dozov [13].

To describe the smectic properties of the TGB phases near h_{c2}, we use a slight modification of the Chen-Lubensky (CL) model [14] for transitions from the nematic (N) to SmA SmC phases. In this model, the smectic molecular number density is expressed as $\rho=\psi+\psi^*$, where ψ is the complex mass-density-wave amplitude with wave numbers with magnitude peaked near $q_0=2\pi/d$. The free energy $F_\psi=F_H+F_{NL}$ is the sum of a nonlinear part, where $F_{NL}=\frac{1}{2}g\int d^3x|\psi|^4$, and a part harmonic in ψ,



(a) (b)

FIG. 3. The square amplitude $|\psi(x,0,0)|^2$ smectic order parameter as a function of x in (a) the Bordeaux phase and (b) the RL TGB$_C$ phase. These figures show the squared slab wave functions $|\phi(x-nl_b)|^2$ with $n=-1$ centered at $-l_b$ (short dash, short space), with $n=1$ centered at l_b (long dash, short space), and with $n=0$ (short and long dashes) and $|\psi(x,0,0)|^2$ (full line). Though $|\phi(x)|$ dies off fairly rapidly, $|\psi(x,0,0)|^2$ has a robust value at grain-boundary positions $x/l_b=\pm0.5$. Note the asymmetry about $x=0$ in the RL case.

$$F_H=\int d^3x\Big[\tilde{r}|\psi|^2+D_{||}|[\nabla_{||}^2(\mathbf{x})+q_0^2]\psi|^2 +D_{\perp}|[\nabla_{\perp}^2(\mathbf{x})+q_{0\perp}^2]\psi|^2 +D_{||\perp}[\nabla_{||}^2(\mathbf{x})+q_0^2]\psi^*[\nabla_{\perp}^2(\mathbf{x})+q_{0\perp}^2]\psi+\text{c.c.}\Big], \quad (1)$$

where $\nabla_{||}^2(\mathbf{x})\equiv[\mathbf{n}(\mathbf{x})\cdot\boldsymbol{\nabla}]^2$ and $\nabla_{\perp}^2(\mathbf{x})=\nabla^2-\nabla_{||}^2(\mathbf{x})$. With the identification $q_{0\perp}^2=-C_{\perp}/2D_{\perp}$ and $\tilde{r}=r-C_{\perp}^2/(4D_{\perp})$, this model is simply an alternative representation of the CL model [14] except for the term proportional to $D_{||\perp}$, which, as we shall see, is needed to stabilize the RL phase. When $q_{0\perp}^2<0$, this model has a phase transition from the N phase to the SmA phase with $\psi=\psi_A e^{iq_0z}$, where $\psi_A=(-r/g)^{1/2}$. When $q_{0\perp}^2>0$, it has a transition to the SmC phase with $\psi=\psi_C e^{i\mathbf{q}_C\cdot\mathbf{x}}$, where $\psi_C=(-\tilde{r}/g)^{1/2}$, with $\mathbf{q}_C=(q_{0\perp}\cos\gamma,q_{0\perp}\sin\gamma,q_0)$ for any angle γ.

To complete the description of our system, we add the Frank free energy $F_{\mathbf{n}}$ for the director including the contribution from molecular chirality,

$$F_{\mathbf{n}}=\frac{1}{2}\int d^3x\{K_1(\boldsymbol{\nabla}\cdot\mathbf{n})^2+K_2[\mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{n})]^2 +K_3[\mathbf{n}\times(\boldsymbol{\nabla}\times\mathbf{n})]^2+h\mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{n})\}. \quad (2)$$

When $\psi=0$, the equilibrium state is the cholesteric phase determined by $F_{\mathbf{n}}$ alone with director

$$\mathbf{n}_0(x)=(0,-\sin k_0x,\cos k_0x), \quad (3)$$

where $k_0=h/K_2\equiv2\pi/P$.

There are several dimensionless quantities in $F=F_\psi+F_{\mathbf{n}}$ that play a role in our analysis. One is the ratio $k_0/q_0=d/P$, which is of the order of 10^{-2} or less. Our primary concern will be the limit in which k_0/q_0 approaches zero, and we will consider only leading terms in this ratio. Other parameters are the ratios

$$\eta_{\perp}=D_{\perp}/D_{||}, \quad \eta_{||\perp}=D_{||\perp}/D_{||}, \quad \omega=q_{\perp0}^2/q_0^2,$$

and the twist Ginzburg parameter, $\kappa_2=(gK_2/2)^{1/2}/(4D_{||}q_0^3)$. In at least one material [15], $\eta_{||\perp}\approx0$ and $\eta_{\perp}\ll1$, but there is no *a priori* reason why either of these conditions should not be violated. $\omega=\tan^2\alpha$ is a measure of the equilibrium tilt angle α between $\mathbf{n}$ and $\mathbf{N}$. It is more convenient to use the twist $k_0=h/K_2$ rather than h as a measure of chirality. The critical twist at which the cholesteric phase becomes unstable to the TGB phases is $k_{c2}=h_{c2}/K_2$.

To determine when the cholesteric phase first becomes unstable with respect to the development of smectic order and to find our variational wave functions [1] for the TGB phases, we calculate the lowest eigenvalues and associated eigenfunctions of the harmonic kernel obtained from F_H with $\mathbf{n}(\mathbf{x})$ replaced by $\mathbf{n}_0(x)$. This kernel K is a periodic function of x with period $P/2$. Its eigenfunctions are, therefore, plane waves in the yz plane that can be expressed as $\psi(\mathbf{x})=\Phi_{\mathbf{q}_P}(x)e^{i\mathbf{q}_P\cdot\mathbf{x}}$, where $\mathbf{q}_P=(0,q_y,q_z)$ and where, as indicated, the form of the function $\Phi_{\mathbf{q}_P}(x)$ can depend on $\mathbf{q}_P$. When $k_0=0$, the eigenfunctions associated with the lowest eigenvalue of K are $\psi(\mathbf{x})=e^{i\mathbf{q}_C\cdot\mathbf{x}}$ in which $\mathbf{q}_C$ can have a nonvanishing x component. We allow explicitly for this component of ψ that varies periodically with x by setting $\Phi_{\mathbf{q}_P}(x)=\phi_{\mathbf{q}_P}(x)e^{iq_x x}$ and $\psi(\mathbf{x})=\phi_{\mathbf{q}_P}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}$, where $\mathbf{q}=(q_x,\mathbf{q}_P)$. With this form for ψ, F_H becomes

$$F_H^0=A\int dx\,\phi_{\mathbf{q}_P}^*(x)\mathcal{H}(x,\hat{\pi},\mathbf{q})\phi_{\mathbf{q}_P}(x), \tag{4}$$

where $\hat{\pi}=i^{-1}d/dx$ is the momentum operator and

$$\begin{aligned}\mathcal{H}(x,\hat{\pi},\mathbf{q})=&\tilde{r}+D_{||}Q_{||}^2(x,\mathbf{q})+D_\perp Q_\perp^2(x,\hat{\pi},\mathbf{q})\\&+D_{||\perp}[Q_{||}(x,\mathbf{q})Q_\perp(x,\hat{\pi},\mathbf{q})\\&+Q_\perp(x,\hat{\pi},\mathbf{q})Q_{||}(x,\mathbf{q})],\end{aligned} \tag{5}$$

where $Q_{||}(x,\mathbf{q})=q_{||}^2(x)-q_0^2$, with $q_{||}^2(x)=[\mathbf{q}\cdot\mathbf{n}_0(x)]^2$, and $Q_\perp(x,\hat{\pi},\mathbf{q})=\hat{\pi}^2+2q_x\hat{\pi}+q_\perp^2(x)-q_{0\perp}^2$, where $q_\perp^2(x)=q^2-q_{||}^2(x)$.

$\mathcal{H}(x,\hat{\pi},\mathbf{q})$ is a periodic function of x with a band spectrum and Bloch eigenfunctions. To the lowest order in k_0/q_0, however, eigenfunctions are localized at spatial minima in $\tilde{r}(x,\mathbf{q})=\mathcal{H}(x,0,\mathbf{q})$, which can be approximated by the lowest-order terms in a Taylor expansion about these minima. For any given $\mathbf{q}$, $\tilde{r}(x,\mathbf{q})$ will have a minimum at some $x=x_m(\mathbf{q})$. Since $\tilde{r}(x,\mathbf{q})$ depends on x only in the combination $\mathbf{n}_0(x)\cdot\mathbf{q}$, $x_m(U(\theta)\mathbf{q})=x_m(\mathbf{q})+\theta/k_0$, where $U(\theta)$ is the operator that rotates $\mathbf{q}_m$ through an angle θ about $\mathbf{p}$, and it is always possible to find a $\mathbf{q}=\mathbf{q}_m$ such that $x_m(\mathbf{q}_m)=0$. If $\phi(x)$ is an eigenfunction of $\mathcal{H}(x,\hat{\pi},\mathbf{q}_m)$ with energy ϵ, then $\psi(\mathbf{x})=\phi(x-\theta/k_0)e^{i\mathbf{q}_m(\theta)\cdot\mathbf{x}}$, where $\mathbf{q}_m(\theta)=U(\theta)\mathbf{q}_m$, is an eigenfunction of the harmonic kernel of F_H with energy ϵ for all θ.

Our approach, therefore, is to find those $\mathbf{q}$'s that minimize $\tilde{r}(0,\mathbf{q})$ or, equivalently, those $\mathbf{q}$'s for which $Q_{||}(0,\mathbf{q})=0$ and $Q_\perp(0,0,\mathbf{q})=0$. Since $q_{||}^2(0)=q_z^2$, and $q_\perp^2(0)=q_x^2+q_y^2$, it follows that $\tilde{r}(0,\mathbf{q})$ is at its minimum equal to $\tilde{r}$ for $q_z=q_0$ and $(q_x,q_y)=q_{0\perp}(\cos\gamma,\sin\gamma)$ for any γ. The Bordeaux phase corresponds to $\gamma=0$, and the RL phase corresponds to $\gamma=\pi/2$. Having found $\mathbf{q}_m$, we can expand $Q_{||}$ and $Q_\perp$ in powers of x and $\hat{\pi}$:

$$\begin{aligned}Q_{||}&=-2k_0q_0q_{0\perp}\sin\gamma x+(q_0^2-q_{0\perp}^2\sin^2\gamma)+\cdots,\\Q_\perp&=\hat{\pi}^2+2q_{0\perp}\cos\gamma\hat{\pi}-Q_{||}.\end{aligned} \tag{6}$$

These expressions simplify in the Bordeaux and RL cases to

$$Q_{||}^B=q_0^2k_0^2x^2,\quad Q_\perp^B=2q_{0\perp}\hat{\pi}-q_0^2k_0^2x^2, \tag{7}$$

$$Q_{||}^{RL}=-2k_0q_0q_{0\perp}x,\quad Q_\perp^{RL}=\hat{\pi}^2-2k_0q_0q_{0\perp}x, \tag{8}$$

plus terms, which we show shortly, that yield corrections to the lowest-order terms in (k_0/q_0). It is clear from these expressions that the B and RL phases enjoy a sort of duality obtained by interchanging x and $\hat{\pi}$. The Hamiltonian for the B phase will have terms proportional to x^4, $\hat{\pi}^2$, and $\hat{\pi}x^2+x^2\hat{\pi}$, whereas that for the RL phase will have terms proportional to $\hat{\pi}^4$, x^2, and $x\hat{\pi}^2+\hat{\pi}^2x$. In the B case, the $\hat{\pi}x^2+x^2\hat{\pi}$ term can be removed by transforming the wave function via $\phi_B(x)=\exp(i\mu_Bx^3)\tilde{\phi}_B(x)$ for an appropriate choice of μ_B, while the $x^2\hat{\pi}+\hat{\pi}x^2$ term in the RL case can be removed by transforming the Fourier transform $\phi_{RL}(k)=\int dxe^{-ikx}\phi_{RL}(x)$ to $\exp(i\mu_{RL}k^3)\tilde{\phi}_{RL}(k)$. In both cases, the eigenfunction $\tilde{\phi}_B(x)$ and $\tilde{\phi}_{RL}(x)$ are localized near $x=0$ over some characteristic length l, and it is convenient to express them as functions of the unitless variable $u=x/l$. This leads to the Hamiltonians for the RL and B phases expressed to the lowest order in u and $\hat{\pi}_u=i^{-1}d/du$:

$$\mathcal{H}_{RL}^0-\tilde{r}=4D_{||}q_0^4s_2\omega(k_0l)^2\left[u^2+\frac{1}{(k_0l)^6}\frac{k_0^4\eta_\perp s_1}{4q_0^4s_2^2\omega}\hat{\pi}_u^4\right],$$

$$\mathcal{H}_B^0-\tilde{r}=D_{||}q_0^4s_1(k_0l)^4\left[u^4+\frac{1}{(k_0l)^6}\frac{4k_0^2\eta_\perp\omega}{q_0^2s_1}\hat{\pi}_u^2\right], \tag{9}$$

where $s_1=1-(\eta_{||\perp}^2/\eta_\perp)$ and $s_2=1+\eta_\perp-2\eta_{||\perp}$. We can choose l to make the respective coefficients of $\hat{\pi}_u^4$ and $\hat{\pi}_u^2$ in $\mathcal{H}_{RL}^0$ and $\mathcal{H}_B^0$ be unity:

$$\begin{aligned}(k_0l_{RL})^6&=(k_0/q_0)^4\eta_\perp s_1/(4s_2^2\omega),\\(k_0l_B)^6&=4(k_0/q_0)^2(\eta_\perp\omega/s_1).\end{aligned} \tag{10}$$

With these choices, $\mathcal{H}_{RL}^0=\tilde{r}+E_0(\eta s_2)^{1/3}(k_0/q_0)^{4/3}[u^2+\hat{\pi}_u^4]$ and $\mathcal{H}_B^0=\tilde{r}+\eta_\perp^{2/3}E_0(k_0/q_0)^{4/3}[u^4+\hat{\pi}_u^2]$, where $E_0=D_{||}q_0^4[16\omega^2s_1]^{1/3}$, are duals to each other with $u^2+\hat{\pi}_u^4$ and $u^4+\hat{\pi}_u^2$ having the same lowest eigenvalue ϵ_0.

The eigenvalues of both $\mathcal{H}_{RL}^0-\tilde{r}$ and $\mathcal{H}_B^0-\tilde{r}$ scale as $\overline{D}q_0^4(k_0/q_0)^{4/3}$. Higher-order terms in k_0x and $\hat{\pi}$ neglected in Eq. (6) yield corrections to the dominant $(k_0/q_0)^{4/3}$ behavior of both $\mathcal{H}_B^0-\tilde{r}$ and $\mathcal{H}_{RL}^0-\tilde{r}$ of the order of $(k_0/q_0)^2$ or higher. In addition $\mathcal{H}^0-\tilde{r}$ scales as k_0/q_0 when $\gamma\neq0,\pi/2$, and corrections to $\mathcal{H}_B^0-\tilde{r}$ and $\mathcal{H}_{RL}^0-\tilde{r}$, respectively, scale as $\gamma^2(k_0/q_0)^{2/3}$ and $(\pi/2-\gamma)^2(k_0/q_0)^{2/3}$. Thus the B and RL phases always have lower energy than phases with intermediate values of γ.

The cholesteric phase becomes unstable at $k_0=k_{c2}$ when the smallest eigenvalue of $\mathcal{H}^0$ becomes zero. Thus

$$k_{c2}^{RL}=\frac{q_0}{(\eta_\perp s_2)^{1/4}}\left(\frac{|\tilde{r}|}{\epsilon_0E_0}\right)^{3/4},\quad k_{c2}^B=\frac{q_0}{\eta_\perp^{1/2}}\left(\frac{|\tilde{r}|}{\epsilon_0E_0}\right)^{3/4},$$

and near $|\tilde{r}|=0$, both $k_{c2}(RL)$ and $k_{c2}(B)$ scale as $|\tilde{r}|^{3/4}$. Their ratio is $k_{c2}^B/k_{c2}^{RL}=(\eta_\perp s_2)^{1/4}$. Thus, $k_{c2}^B>k_{c2}^{RL}$ and the cholesteric phase becomes unstable to the B phase before the RL phase when $s_2=1+\eta_\perp-2\eta_{||\perp}>\eta_\perp$, i.e., when $\eta_{||\perp}<1/2$, and it becomes unstable to the RL phase before the B phase when $\eta_{||\perp}>1/2$. This means that the RL phase is not stable in the original CL model in which $D_{||\perp}=0$. Stability of the nematic phase in the absence of chirality requires $D_{||}D_\perp-D_{||\perp}^2=D_{||}D_\perp s_1>0$ or $\eta_{||\perp}^2<\eta_\perp$. It is clearly pos-

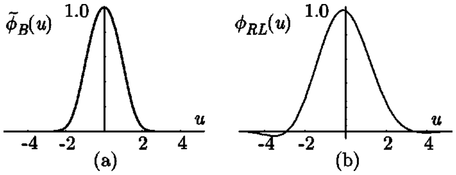


FIG. 4. Amplitude of eigenfunctions for (a) the Bordeaux case and (b) the RL case. Note that the RL wave function has an oscillatory component and it is slightly asymmetric.

sible to satisfy both this condition and $\eta_{||\perp} > 1/2$, so that there is a range of parameters for which the RL phase is stable.

The B eigenfunctions are of the form $\phi(u) = e^{i\mu'_B u^3}\tilde{\phi}_B(u)$, where $(u^4 + \hat{\pi}_u^2)\tilde{\phi}_B(u) = \epsilon_0 \tilde{\phi}_B(u)$. The RL eigenfunctions are of the form $\phi_{RL}(u) = \int (dk/2\pi) e^{-i\mu'_{RL}k^3}\tilde{\phi}_B(k)$, where $\tilde{\phi}_B(u)$ and $\tilde{\phi}_B(k)$ are identical functions of different arguments. $\tilde{\phi}_B(u)$ can be obtained numerically using the shooting method, and it, along with $\phi_{RL}(u)$, is plotted in Fig. 4. The resulting eigenvalue is $\epsilon_0 = 1.060\,357\ldots$. An excellent approximation to $\phi_B(u)$ over the entire range of u is $\tilde{\phi}(u) = \exp[-\frac{1}{2}Au^2\sqrt{1+\frac{4}{9}(u/A)^2}]$, where $A = 1.035$. This function satisfies the requirement that $\tilde{\phi}(u) \to \exp(-\frac{1}{3}|u^3|)$ as $|u| \to \infty$.

To determine the structure of the TGB$_C$ phases, we construct variational smectic order parameters from the degenerate set of lowest eigenfunctions of $\mathcal{H}(x, \hat{\pi}, \mathbf{q})$,

$$\psi(\mathbf{x}) = C \sum_s \phi[(x - nl_b)/l] e^{iq_m(\theta)\cdot\mathbf{x}}, \qquad (11)$$

where $\mathbf{q}_m(\theta) = U(\theta)\mathbf{q}_m$ and, as discussed in the paragraph preceding Eq. (9), $\mathbf{q}_m$ has different forms in the Bordeaux and RL phases. Following the treatment of the cholesteric-to-TGB$_A$ transition [1], we can write the total free energy after minimizing over director fluctuations as

$$\frac{F}{K_2 q_0^2 V} = -\frac{1}{2} - \frac{\mathcal{A}}{4\beta(l_b/l)}\left(\frac{k_0 - k_{c2}}{k_0}\right)^2, \qquad (12)$$

where $\beta(l_b/l)$ depends on the separation l_b between grain boundaries, and $\mathcal{A}$ depends on k_0 and l but not on l_b. Thus, the equilibrium value of l_b is that which minimizes $\beta(l_b/l)$, which can be expressed as $\{[\psi^4]_{\mathrm{av}} - \kappa_2^{-2}(k_0/q_0)^{2/3}[f]_{\mathrm{av}}\}/[\psi^2]_{\mathrm{av}}^2$, where f is a complicated function of the order of ψ^4 and $[g]_{\mathrm{av}} = V^{-1}\int d^3xg$ is the spatial average of g. Carrying out this minimization procedure using the analytic approximation for $\phi_B(u)$, we find $l_b/l \approx 2.15$ for the Bordeaux and $l_b/l \approx 3.0$ for the RL TGB$_C$ phases, respectively. To find the spacing between dislocations in a grain boundary, we use the geometric relation $k_0 = d/(\sin\alpha_0 l_b l_d)$, where $\pi/2 - \alpha_0$ is the angle between $\mathbf{N}$ and $\mathbf{p}$. Our results for $l_b \sim l$, $l_d \sim Pd/l$, and $\xi \sim (Dq_0^2/\tilde{r})^{1/2} \sim d^{1/3}P^{2/3}$ are summarized in Table I. The wave function ψ for our calculated values of l_b for both TGB$_C$ phases are shown in Fig. 3.

We have presented an overview of the properties of the the Bordeaux and RL TGB$_C$ phases and the transition to them from the cholesteric phase obtained from an Abrikosov-like analysis near the upper critical twist k_{c2} at which the cholesteric phase becomes unstable. In a future paper [16], we will present more details of our calculations. We will also discuss the relation between our work and that of Dozov [13].

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